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DEVELOPMENT OF METHODS FOR THE STABILITY ANALYSIS OF POSITIVE SYSTEMS

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Abstract

A review of methods for the stability analyzis of positive dynamical systems is provided.

A set of results concerning stability analysis of some classes of linear and nonlinear positive systems is presented.

Both switched and nonswitched systems are considered. Delay-free and time-delay systems are studied.

The focus is on the problem of **diagonal stability** (constructing diagonal Lyapunov functions and Lyapunov–Krasovskii functionals).

Applications of the presented approaches to problems of population dynamics and control of mobile agent formations are considered.

Notation

Let $\mathbb{R}$, $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$ denote the field of real numbers, the n-dimensional Euclidean space and the vector space of $n \times n$ matrices, respectively.

Let $\|\cdot\|$ be the Euclidean norm of a vector.

For a given number $\tau > 0$, $C([-\tau, 0], \mathbb{R}^n)$ is the space of continuous functions $\varphi : [-\tau, 0] \mapsto \mathbb{R}^n$ with the uniform norm

$$\|\varphi\|_\tau = \max_{\xi\in[-\tau,0]} \|\varphi(\xi)\|.$$

A matrix $A \in \mathbb{R}^{n\times n}$ is **Hurwitz** if all its eigenvalues have negative real parts.

A matrix $A \in \mathbb{R}^{n\times n}$ is **Schur-Cohn** if all its eigenvalues have modules less than one.

A matrix $A \in \mathbb{R}^{n\times n}$ is called **nonnegative** if all its entries are nonnegative.

A matrix $A \in \mathbb{R}^{n\times n}$ is called **Metzler** if all its off-diagonal entries are nonnegative.

The identity matrix is denoted by I.

Let diag$\{\lambda_1, \dots, \lambda_n\}$ be the **diagonal matrix** with the elements $\lambda_1, \dots, \lambda_n$ along the main diagonal.

For a vector $v \in \mathbb{R}^n$, $v \geq 0$ means that $v_i \geq 0$ for $i = 1, \ldots, n$. Similarly, $v > 0$ means that $v_i > 0$ for $i = 1, \ldots, n$.

For a vector $x \in \mathbb{R}^n$, $|x|$ denotes the vector $(|x_1|, \ldots, |x_n|)^T$.

For a symmetric matrix M, $M \succ 0$ ($M \prec 0$) denotes that M is positive (negative) definite.

A matrix A is called **diagonally stable** if there exists a positive definite diagonal matrix $D \in \mathbb{R}^{n \times n}$ such that $A^T D + DA \prec 0$.

1. Delay-Free Systems

1.1. Linear positive systems

Consider the system

$$\dot{x}(t) = Ax(t). \tag{1}$$

Here $x(t) \in \mathbb{R}^n$ is the state vector, and A is a constant matrix.

Definition 1 [Farina, Rinaldi, 2000]: System (1) is called positive if $x_0 \geq 0$ implies $x(t, t_0, x_0) \geq 0$ for all $t_0 \geq 0$ and $t \geq t_0$.

Lemma 1 [Farina, Rinaldi, 2000]: System (1) is positive if and only if A is a Metzler matrix.

Lemma 2 [Kaszkurewicz, Bhaya, 2000]: Let A be Metzler matrix. The following are equivalent:
(i) A is Hurwitz;
(ii) there exists a vector $v > 0$ in $\mathbb{R}^n$ with

$$A^\top v < 0; \tag{2}$$

(iii) there exists a vector $\theta > 0$ in $\mathbb{R}^n$ with

$$A\theta < 0; \tag{3}$$

(iv) matrix A is diagonally stable;
(v) $\det A \neq 0$ and $A^{-1} \leq 0$.

Lyapunov functions for (1)

Theorem 1 [Farina, Rinaldi, 2000]: If A is Metzler and Hurwitz matrix, then the system (1) admits the following Lyapunov functions:

$$V_1 = v^\top |x| = \sum_{i=1}^{n} v_i |x_i|, \tag{4}$$

where v is a positive vector satisfying (2);

$$V_2 = \max_{i=1,\dots,n} \frac{|x_i|}{\theta_i}, \tag{5}$$

where θ_i are components of a positive vector θ satisfying (3);

$$V_3 = \sum_{i=1}^{n} \lambda_i x_i^2, \tag{6}$$

where $\lambda_i = v_i/\theta_i$ and v_i, θ_i are components of positive vectors v, θ satisfying (2) and (3), respectively.

1.2. Switched linear positive systems

Consider the switched system

$$\dot{x}(t) = A_\sigma x(t) \tag{7}$$

and the associated family of subsystems

$$\dot{x}(t) = A_s x(t), \quad s = 1, \ldots, N, \tag{8}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $A_1, \ldots, A_N$ are constant matrices, switching law $\sigma = \sigma(t)$ is a piecewise constant function $\sigma : [0, \infty) \to \{1, \ldots, N\}$, which has only finitely many discontinuities on every bounded interval.

Lemma 3. The system (7) is positive if and only if $A_1, \ldots, A_N$ are Metzler matrices.

Consider the families of linear algebraic inequalities

$$A_s^\top v < 0, \quad s = 1, \dots, N, \tag{9}$$

$$A_s \theta < 0, \quad s = 1, \dots, N. \tag{10}$$

Theorem 2. Let $A_1, \dots, A_N$ be Metzler matrices. Then

(i) if there exists a positive vector v satisfying (9), then the family (8) admits the common Lyapunov function

$$V_1 = v^\top |x|;$$

(ii) if there exists a positive vector θ satisfying (10), then the family (8) admits the common Lyapunov function

$$V_2 = \max_{i=1,\dots,n} \frac{|x_i|}{\theta_i};$$

(iii) if there exist positive vectors v and θ satisfying (9) and (10), respectively, then the family (8) admits the common Lyapunov function

$$V_3 = \sum_{i=1}^{n} \lambda_i x_i^2, \qquad \lambda_i = v_i/\theta_i.$$

Theorem 3 [Pastravanu, Matcovschi, 2014]. Let $A_1, \dots, A_N$ be Metzler matrices. If there exist vectors $v > 0$, $\theta > 0$ and numbers μ, ω such that

$$A_s^\top v \le \mu v, \quad s = 1, \dots, N, \tag{11}$$

$$A_s \theta \le \omega \theta, \quad s = 1, \dots, N, \tag{12}$$

$$\mu + \omega < 0, \tag{13}$$

the family (8) admits the common Lyapunov function

$$V_3 = \sum_{i=1}^{n} \lambda_i x_i^2, \qquad \lambda_i = v_i/\theta_i.$$

1.3. Persidskii-type switched systems

Consider the switched system

$$\dot{x}(t) = A_\sigma f(x(t)) \tag{14}$$

and the corresponding family of subsystems

$$\dot{x}(t) = A_s f(x(t)), \quad s = 1, \ldots, N. \tag{15}$$

Here $x(t) \in \mathbb{R}^n$ is the state vector, and $A_1, \ldots, A_N$ are constant matrices;

the diagonal nonlinearity $f(x) = (f_1(x_1), \ldots, f_n(x_n))^\top$ is continuous for $x \in \mathbb{R}^n$ and satisfies the sector condition

$$x_i f_i(x_i) > 0 \quad \text{for} \quad x_i \neq 0, \quad i = 1, \ldots, n. \tag{16}$$

Hence, the system (14) admits the zero solution.

The subsystems from the family (15) are well-known **Persidskii-type systems** [Persidskii, 1970; Kaszkurewicz, Bhaya, 2000]. They are widely used for modeling automatic control systems, neural networks, biological systems, end etc.

Definition [Aleksandrov, Platonov, 2008]: The system (14) is called absolutely stable if its zero solution is asymptotically stable for any admissible nonlinearities and any admissible switching law.

Assumption 1. Let $A_1, \ldots, A_N$ be Metzler matrices.

Under Assumption 1, the system (14) is positive.

Theorem 4 [Aleksandrov, Platonov, 2008]: If there exists a vector $v > 0$ satisfying the inequalities

$$A_s^\top v < 0, \qquad s = 1, \dots, N, \tag{17}$$

then
$$V_1 = v^\top |x| \tag{18}$$
is the common linear Lyapunov function for the family (15).

Theorem 5 [Aleksandrov, Platonov, 2008]: Let the functions $f_i(x_i)$ be continuously differentiable for $|x_i| < \Delta$ ($\Delta = \text{const} > 0$) and $f_i'(x_i) > 0$ for $0 < |x_i| < \Delta$. If there exists a vector $\theta > 0$ satisfying the inequalities

$$A_s \theta < 0, \qquad s = 1, \dots, N, \tag{19}$$

then

$$V_2 = \max_{i=1,\dots,n} \frac{|f_i(x_i)|}{\theta_i} \tag{20}$$

is the common max-type Lyapunov function for the family (15).

Theorem 7 [Aleksandrov, Chen, Platonov, Zhang, 2011]: If there exist vectors $\eta > 0$ and $\zeta > 0$ satisfying (17) and (19), respectively, then, for any rational $\gamma > 0$ with odd numerator and denominator, then the family (15) admits a common diagonal Lyapunov function of the form

$$V_3 = \sum_{i=1}^{n} \lambda_i \int_0^{x_i} f_i^{\gamma}(u)du. \tag{21}$$

Here $\lambda_i = v_i/\theta_i^{\gamma}$.

Theorem 8 [Aleksandrov, 2021]: Let γ be a given positive rational number with odd numerator and denominator. If there exist vectors $v > 0$, $\theta > 0$ and numbers μ, ω such that

$$\mu + \gamma\omega < 0, \quad A_s^{\top} v \leq \mu v, \quad A_s\theta \leq \omega\theta, \quad s = 1, \dots, N, \tag{22}$$

then the family (15) admits the common Lyapunov function (21) with $\lambda_i = v_i/\theta_i^{\gamma}$.

Corollary 1 [Aleksandrov, 2021]: Let there exist a positive vector θ satisfying the inequalities

$$A_s\theta < 0, \qquad s = 1, \ldots, N. \tag{23}$$

Then one can choose a number $\gamma_0 > 0$ such that, for any rational $\gamma > \gamma_0$ with odd numerator and denominator, the family (15) admits a common Lyapunov function of the form (21).

Corollary 2 [Aleksandrov, 2021]: Let there exist a positive vector v satisfying the inequalities $$A_s^\top v < 0, \qquad s = 1, \ldots, N. \tag{24}$$
Then one can choose a number $\gamma_0 > 0$ such that, for any rational $\gamma \in (0, \gamma_0)$ with odd numerator and denominator, the family (15) admits a common Lyapunov function of the form (21).

Remark. The function (21) provides us the continuous connection between linear and max-type Lyapunov functions as $\gamma \in (0, +\infty)$.

2. Time-Delay Systems

2.1. Linear time-invariant positive systems

$$\dot{x}(t) = Ax(t) + Bx(t - \tau). \tag{1}$$

Here $x(t) \in \mathbb{R}^n$, A and B are constant matrices, $\tau = \text{const} \geq 0$ is a delay.

Definition [Smith, 1995]: The system (1) is called positive if $\varphi(\theta) \geq 0$ for $\theta \in [-\tau, 0]$ implies $x(t, t_0, \varphi) \geq 0$ for all $t_0 \geq 0$ and $t \geq t_0$.

Lemma 1 [Smith, 1995]: The system (1) is positive if and only if A is a Metzler matrix and B is a nonnegative matrix.

Lemma 2 [Smith, 1995]: The positive system (1) is asymptotically stable for any $\tau \geq 0$ if and only if $A + B$ is a Hurwitz matrix.

2.2. Conditions of diagonal Riccati stability

A Lyapunov–Krasovskii functional for (1):

$$V = x^{\top}(t)Px(t) + \int_{-\tau}^{0} x^{\top}(t+\xi)Qx(t+\xi)d\xi, \tag{2}$$

where P and Q are constant positive definite matrices.

It is known [Verriest, 2004] that the derivative of (2) along the solutions of (1) is negative definite if and only if the Riccati inequality

$$A^{\top}P + PA + Q + PBQ^{-1}B^{\top}P \prec 0 \tag{3}$$

holds.

Definition [Mason, 2012]: The pair of matrices A, B is called diagonally Riccati stable if there exist positive definite **diagonal** matrices P and Q satisfying (3).

Theorem 1 [Aleksandrov, Mason, 2016]: Let A be Metzler matrix, and B be nonnegative matrix. The pair A, B is diagonally Riccati stable if and only if $A + B$ is Hurwitz matrix.

Corollary [Aleksandrov, Mason, 2016]: Let A be Metzler matrix, and B be nonnegative matrix. System (1) is asymptotically stable for any $\tau \geq 0$ if and only if the pair A, B is diagonally Riccati stable.

Remark. In [Aleksandrov, Mason, 2016], constructive approach to finding positive definite diagonal matrices P and Q satisfying (3) is proposed.

Remark. Some conditions of diagonal Riccati stability for the case where (1) is not a positive system were obtained in [Aleksandrov, Mason, 2016], [Aleksandrov, Mason, Vorob'eva, 2017], [Shen, Lam, 2020], etc.

2.3. Persidskii-type systems with delay

$$\dot{x}(t) = Af(x(t)) + Bf(x(t-\tau)). \tag{4}$$

Here $x(t) \in \mathbb{R}^n$, A and B are constant matrices, τ is a constant nonnegative delay, the diagonal nonlinearity $f(x) = (f_1(x_1), \ldots, f_n(x_n))^\top$ is continuous for $x \in \mathbb{R}^n$ and satisfies the sector condition

$$x_i f_i(x_i) > 0 \quad \text{for} \quad x_i \neq 0, \quad i = 1, \ldots, n. \tag{5}$$

Hence, the system (4) admits the zero solution.

Definition [Aleksandrov, Mason, 2018]: The system (4) is called absolutely stable if its zero solution is asymptotically stable for any admissible nonlinearities and any nonnegative delay.

Theorem 2 [Aleksandrov, Mason, 2016]: Let A be Metzler matrix and B be nonnegative matrix. The system (4) is absolutely stable if and only if $A+B$ is Hurwitz matrix.

Theorem 3 [Aleksandrov, Mason, 2016]: Let A be Metzler matrix and B be nonnegative matrix. The system (4) is absolutely stable if and only if for the system there exists a diagonal Lyapunov-Krasovskii functional of the form

$$V = \sum_{i=1}^{n} \nu_i \int_0^{x_i(t)} f_i(\zeta) d\zeta + \sum_{j=1}^{n} \mu_j \int_{t-\tau}^{t} f_j^2(x_j(u)) du, \qquad (6)$$

where ν_i and μ_j are positive coefficients.

2.4. Generalised Lotka-Volterra Systems

Let us show that the results described above can be applied to a generalised class of Lotka-Volterra models occurring in population dynamics.

Consider the system

$$\dot{x}_i(t) = g_i(x_i(t)) \left(c_i + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau)) \right), \tag{7}$$

$$i = 1, \dots, n.$$

Here $x_i(t)$ is the population density of the i-th species; the functions $g_i : [0, +\infty) \to [0, +\infty)$ and $f_i : [0, +\infty) \to [0, +\infty)$ possess special properties described below; c_i, a_{ij}, b_{ij} are constant coefficients.

The system (7) describes the interaction n species in a biological community. It is a generalization of the classical Lotka–Volterra model, see [Hofbauer, Sigmund, 1998].

The coefficients c_i characterise the intrinsic growth rate of the i-th population; the self-interaction terms $a_{ii}g_i(x_i)f_i(x_i)$ with $a_{ii} < 0$ reflect the limited resources available in the environment; the terms $a_{ij}g_i(x_i)f_j(x_j)$ and $b_{ij}g_i(x_i(t))f_j(x_j(t-\tau))$ for $j \neq i$ describe the influence of population j on population i.

Let $\mathbb{R}^n_+$ be the nonnegative cone of $\mathbb{R}^n$: $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \mid x \geq 0\}$.

Also, $\operatorname{int}\mathbb{R}^n_+$ is the interior of $\mathbb{R}^n_+$.

We consider functions f_i and g_i satisfying the following conditions:

(i) g_i and f_i are continuous on $[0, +\infty)$;
(ii) $g_i(0) = f_i(0) = 0$, $g_i(x_i) > 0$, $f_i(x_i) > 0$ for $x_i > 0$;
(iii) f_i is a strictly increasing function on $[0, +\infty)$, and $f_i(x_i) \to +\infty$ as $x_i \to +\infty$;
(iv) $\int_1^{+\infty} f_i(\zeta)/g_i(\zeta)\, d\zeta = +\infty$;
(v) $\int_0^1 1/g_i(\zeta)\, d\zeta = +\infty$.

The assumptions on the system imply that $\operatorname{int}\mathbb{R}^n_+$ is an invariant set for (7). For biological reasons, we will consider (7) with respect to the state space $\operatorname{int}\mathbb{R}^n_+$.

Let $A = \{a_{ij}\}_{i,j=1}^n$, $B = \{b_{ij}\}_{i,j=1}^n$, and $c = (c_1, \ldots, c_n)^\top$.

Theorem 4 [Aleksandrov, Mason, 2016]: Assume that A is Metzler, B is nonnegative and $c > 0$. If $A + B$ is Hurwitz matrix, then the system (7) admits an unique equilibrium point $\bar{x}$ in $\operatorname{int} \mathbb{R}^n_+$ that is globally asymptotically stable in $\operatorname{int} \mathbb{R}^n_+$ for all $\tau \geq 0$.

Really, as $A + B$ is Metzler and Hurwitz, $(A + B)^{-1} \leq 0$. Hence,

$$-(A + B)^{-1} c > 0. \tag{8}$$

It follows from property (iii) of the functions f_i, $1 \leq i \leq n$, that there exists an unique $\bar{x} \in \operatorname{int} \mathbb{R}^n_+$ satisfying

$$f(\bar{x}) = -(A + B)^{-1} c,$$

which implies that $\bar{x}$ is an equilibrium of (7).

Then system (7) can be rewritten as follows

$$\dot{x}_i(t) = g_i(x_i(t)) \left(\sum_{j=1}^{n} a_{ij}(f_j(x_j(t)) - f_j(\bar{x}_j)) \right.$$

$$\left. + \sum_{j=1}^{n} b_{ij}(f_j(x_j(t-\tau)) - f_j(\bar{x}_j)) \right), \quad i = 1, \ldots, n.$$

From the properties of matrices A and B it follows that for this systam there exists a **diagonal** Lyapunov-Krasovskii functional of the form

$$V = \sum_{i=1}^{n} \nu_i \int_{\bar{x}_i}^{\varphi(0)} \frac{f_i(\zeta) - f_i(\bar{x}_i)}{g_i(\zeta)} d\zeta + \sum_{j=1}^{n} \mu_j \int_{-\tau}^{0} (f_j(\varphi_j(u)) - f_j(\bar{x}_j))^2 du, \quad (9)$$

where ν_i and μ_j are positive coefficients.

2.5. Discrete-Time Positive Systems

A Linear Positive System. Consider the system

$$x(k+1) = Ax(k) + B_1x(k-1) + \ldots + B_lx(k-l), \tag{10}$$

where $A, B_1, \ldots, B_l$ are nonnegative matrices in $\mathbb{R}^{n\times n}$.

Under this assumption (10) defines a positive time-delay system in discrete time.

Theorem 5 [Aleksandrov, Mason, 2014]: Let the matrix

$$S = A + B_1 + \ldots + B_l \tag{11}$$

is Schur-Cohn. Then there exists a Lyapunov-Krasovskii functional for (10) of the form

$$\begin{gathered} V = x^T(k)Px(k) + x^T(k-1)Q_1x(k-1) \\ +\{x^T(k-1)Q_2x(k-1) + x^T(k-2)Q_2x(k-2)\} + \cdots \\ +\{x^T(k-1)Q_lx(k-1) + \cdots + x^T(k-l)Q_lx(k-l)\} \end{gathered} \tag{12}$$

where $P, Q_1, \ldots, Q_l$ are positive definite **diagonal** matrices.

A Nonlinear Positive System. Consider the system

$$x(k+1) = Af(x(k)) + B_1 f(x(k-1)) + \ldots + B_l f(x(k-l)) \qquad (13)$$

where $x(k) \in \mathbb{R}^n$, $A, B_1, \ldots, B_l$ are nonnegative matrices, and the nonlinearity $f(x) = (f_1(x_1), \ldots, f_n(x_n))^\top$ is continuous for $x \in \mathbb{R}^n$ and such that

$$x_i f_i(x_i) > 0 \quad \text{for} \quad x_i \neq 0, \quad |f_i(x_i)| \leq |x_i|, \quad i = 1, \ldots, n, \qquad (14)$$

Theorem 6 [Aleksandrov, Mason, 2014]: Let S given by (11) be a Schur-Cohn matrix. Then there exists a Lyapunov-Krasovskii functional of the form

$$V = x^T(k)Px(k) + f^T(x(k-1))Q_1 f(x(k-1))$$

$$+\{f^T(x(k-1))Q_2 f(x(k-1)) + f^T(x(k-2))Q_2 f(x(k-2))\} + \cdots$$

$$+\{f^T(x(k-1))Q_l f(x(k-1)) + \ldots + f^T(x(k-l))Q_l f(x(k-l))\} \qquad (15)$$

where $P, Q_1, \ldots, Q_l$ are positive definite **diagonal** matrices.

2.6. Switched Time-Delay Persidskii-type Systems

$$\dot{x}(t) = A_\sigma f(x(t)) + B_\sigma f(x(t-\tau)). \tag{16}$$

The corresponding family of subsystems

$$\dot{x}(t) = A_s f(x(t)) + B_s f(x(t-\tau)), \quad s = 1, \ldots, N. \tag{17}$$

Here $x(t) \in \mathbb{R}^n$, A_s and B_s are constant matrices, $\tau \geq 0$ is a constant delay, $f(x)$ is a continuous for $x \in \mathbb{R}^n$ diagonal nonlinearity satisfying the sector condition $x_i f_i(x_i) > 0$ for $x_i \neq 0$, $i = 1, \ldots, n$.
Hence, the system (16) admits the zero solution.

We will assume that A_s are Metzler matrices, and $B_s \geq 0$. Then (16) is a positive system.

Definition [Aleksandrov, Mason, 2014]: The system (16) is called absolutely stable if its zero solution is asymptotically stable for any admissible nonlinearities, any admissible switching law and for $\tau \geq 0$.

Conditions of the existence of a common linear functional

Theorem 7 [Aleksandrov, Mason, 2014]: Assume that $A^{(s)}$ is Metzler and $B^{(s)}$ is nonnegative for $s = 1, \ldots, N$. If there exists a vector $v > 0$ such that

$$(A^{(s)} + B^{(r)})^\top v < 0, \qquad s, r = 1, \ldots, N, \tag{18}$$

then system (16) is absolutely stable, and there exist positive numbers $\mu_1, \ldots, \mu_n$ such that

$$V = \sum_{i=1}^{n} v_i |x_i(t)| + \sum_{i=1}^{n} \mu_i \int_{t-\tau}^{t} |f_i(x_i(z))| dz \tag{19}$$

defines a common Lyapunov–Krasovskii functional for the family (17).

Remark: In [Aleksandrov, Mason, 2014], Theorem 7 was extended to:

Switched Persidskii-type Systems with Multiple Delays
Coupled Nonlinear Differential and Difference Systems
Switched Nonlinear Difference Systems with Time-Delay
Neutral Type Systems

Conditions of the existence of a common diagonal functional

Theorem 8 [Aleksandrov, Mason, 2018]: Let $A^{(s)}$ be Metzler and $B^{(s)}$ be nonnegative matrices for $s = 1, \dots, N$. If there exist positive vectors v and θ such that

$$(A^{(s)} + B^{(r)})^\top v < 0, \quad s, r = 1, \dots, N, \tag{20}$$

$$(A^{(s)} + B^{(s)})\theta < 0, \quad s = 1, \dots, N, \tag{21}$$

then the system (16) is absolutely stable, and the family (17) admits a common diagonal Lyapunov–Krasovskii functional of the form

$$V = \sum_{i=1}^{n} \lambda_i \int_0^{x_i(t)} f_i(\zeta) d\zeta + \sum_{j=1}^{n} \mu_j \int_{t-\tau}^{t} f_j^2(x_j(u)) du, \tag{22}$$

where λ_i and μ_j are positive coefficients.

Theorem 9 [Aleksandrov, Mason, 2018]: Let $A^{(s)}$ be Metzler and $B^{(s)}$ be nonnegative matrices for $s = 1, \ldots, N$. If there exist numbers α, β and positive vectors v and θ such that

$$(A^{(s)} + B^{(r)})^\top v \le \alpha v, \quad s, r = 1, \ldots, N, \tag{23}$$

$$(A^{(s)} + B^{(s)})\theta \le \beta\theta, \quad s = 1, \ldots, N, \tag{24}$$

$$\alpha + \beta < 0, \tag{25}$$

then the system (16) is absolutely stable, and the family (17) admits a common diagonal Lyapunov–Krasovskii functional of the form (22).

Remark. In [Aleksandrov, Mason, 2018], a constructive approach for finding coefficients λ_i and μ_j of the functional (22) was proposed.

Example [Aleksandrov, Mason, 2018]: Consider the automatic control system with switching and delay in the feedback law

$$\begin{aligned} \dot{x}_i(t) &= a_i^{(\sigma)} x_i(t) + b_i^{(\sigma)} \varphi(x_n(t-\tau)), \quad i = 1, \dots, n-1, \\ \dot{x}_n(t) &= \sum_{j=1}^{n-1} c_j^{(\sigma)} x_j(t) + a_n^{(\sigma)} \varphi(x_n(t)), \end{aligned} \tag{26}$$

and the corresponding family of subsystems

$$\begin{aligned} \dot{x}_i(t) &= a_i^{(s)} x_i(t) + b_i^{(s)} \varphi(x_n(t-\tau)), \quad i = 1, \dots, n-1, \\ \dot{x}_n(t) &= \sum_{j=1}^{n-1} c_j^{(s)} x_j(t) + a_n^{(s)} \varphi(x_n(t)), \quad s = 1, \dots, N. \end{aligned} \tag{27}$$

Here $\sigma = \sigma(t)$ is an addmissible switching law, $\sigma(t) : [0, +\infty) \mapsto \{1, \dots, N\}$, $a_j^{(s)}, c_i^{(s)}, b_i^{(s)}$ are constant coefficients with $a_j^{(s)} < 0$, $c_i^{(s)} > 0$, $b_i^{(s)} > 0$, $i = 1, \dots, n-1$, $j = 1, \dots, n$, $s = 1, \dots, N$;

$\varphi(x_n)$ is a scalar nonlinearity that is continuous for $x_n \in (-\infty, +\infty)$ and satisfies the sector condition $x_n\varphi(x_n) > 0$ for $x_n \neq 0$;
τ is a constant nonnegative delay.

The system (26) is a special case of (16). Here $f_i(x_i) = x_i$, $i = 1, \ldots, n-1$, $f_n(x_n) = \varphi(x_n)$.

Applying Theorem 8 we obtain that if

$$\sum_{i=1}^{n-1} c_i^{(s)} \max_{l=1,\ldots,N} \frac{b_i^{(l)}}{|a_i^{(l)}|} < |a_n^{(s)}|, \quad s = 1, \ldots, N, \tag{28}$$

$$\sum_{i=1}^{n-1} b_i^{(r)} \max_{l=1,\ldots,N} \frac{c_i^{(l)}}{|a_i^{(l)}|} + \max_{s=1,\ldots,N} a_n^{(s)} < 0, \quad r = 1, \ldots, N, \tag{29}$$

then the system (26) is absolute stability and the family (27) admits a common diagonal Lyapunov–Krasovskii functional of the form (22).

Theorem 9 provides us less conservative conditions:

$$\lambda+\mu<0, \qquad \lambda > \max_{j=1,\dots,n} \max_{s=1,\dots,N} a_j^{(s)}, \qquad \mu > \max_{j=1,\dots,n} \max_{s=1,\dots,N} a_j^{(s)}, \tag{30}$$

$$\sum_{i=1}^{n-1} c_i^{(s)} \max_{l=1,\dots,N} \frac{b_i^{(l)}}{\lambda - a_i^{(l)}} \le \lambda - a_n^{(s)}, \quad s=1,\dots,N, \tag{31}$$

$$\sum_{i=1}^{n-1} b_i^{(r)} \max_{l=1,\dots,N} \frac{c_i^{(l)}}{\mu - a_i^{(l)}} + \max_{s=1,\dots,N} a_n^{(s)} \le \mu, \quad r=1,\dots,N. \tag{32}$$

Theorem 10 [Aleksandrov, 2024]: Let $A^{(s)}$ be Metzler and $B^{(s)}$ be nonnegative matrices for $s = 1, \dots, N$, and a positive rational γ with odd numerator and denominator be given. If there exist numbers α, β and positive vectors v and θ such that

$$\gamma\beta + \alpha < 0, \tag{33}$$

$$(A^{(s)} + B^{(r)})^\top v \le \alpha v, \quad s, r = 1, \dots, N, \tag{34}$$

$$(A^{(s)} + B^{(s)})\theta \le \beta\theta, \quad s = 1, \dots, N, \tag{35}$$

then the family (16) admits a common Lyapunov–Krasovskii functional of the form

$$V = \sum_{i=1}^{n} \lambda_i \int_0^{x_i(t)} f_i^{\gamma}(\zeta) d\zeta + \sum_{j=1}^{n} \mu_j \int_{t-\tau}^{t} f_j^{\gamma+1}(x_j(u)) du, \tag{36}$$

where λ_i and μ_j are positive coefficients, ensuring the absolute stability of (17).

Some extensions of the results

1. **Distributed delay:**

$$\dot{x}(t) = A_\sigma f(x(t)) + B_\sigma \int_{t-\tau}^{t} g(t-\xi) f(x(\xi)) d\xi.$$

2. **Unbounded delay:**

$$\dot{x}(t) = A_\sigma f(x(t)) + B_\sigma \int_{0}^{t} g(t-\xi) f(x(\xi)) d\xi.$$

3. **Infinite delay:**

$$\dot{x}(t) = A_\sigma f(x(t)) + B_\sigma \int_{-\infty}^{t} g(t-\xi) f(x(\xi)) d\xi.$$

4. Functional difference systems:

$$x(t) = A_\sigma x(t-h) + B_\sigma \int_{t-\tau}^{t} x(\xi)d\xi.$$

5. Coupled systems:

$$\begin{cases} \dot{x}(t) = A_\sigma f(x(t)) + B_\sigma y(t-\tau) + C_\sigma \int_{t-\beta}^{t} y(\xi)d\xi, \\ y(t) = P_\sigma f(x(t)) + G_\sigma y(t-\tau) + Q_\sigma \int_{t-\beta}^{t} y(\xi)d\xi. \end{cases}$$

6. Permanence conditions for switched generalized Lotka–Volterra models.

2.7. Extensions to Nonlinear Wazewski's Autonomous Systems

$$\dot{x}(t) = Ax(t) \qquad \rightarrow \qquad \theta > 0, \quad A\theta < 0.$$

A.A. Martynyuk, A.Yu. Obolenskii, "Stability of solutions of Wazewski's autonomous systems," Diff. Uravn., 16, No. 8, 1392-1407 (1980).

MO-condition:

$$\dot{x}(t) = F(x(t)) \qquad \rightarrow \qquad \theta > 0, \quad F(\theta) < 0.$$

An extension to nonlinear autonomous Wazewski systems with delay:
Obolenskii, A.Y. ”Stability of solutions of autonomous Wazewski systems with delayed action.” Ukr. Math. J. 35, 486-492 (1983).

Review paper:
A.A. Martynyuk. ”ASYMPTOTIC STABILITY CRITERION FOR NONLINEAR MONOTONIC SYSTEMS AND ITS APPLICATIONS (REVIEW)”, International Applied Mechanics, Vol. 47, No. 5, 2011.

Applications of MO-condition to stability analysis of generalized Persidskii-type systems and nonlinear complex systems:
A. Yu. Aleksandrov, A. V. Platonov, ”Aggregation and stability analysis of nonlinear complex systems,” J. Math. Anal. Appl. 342 (2008) 989-1002.
A. Yu. Aleksandrov, A. V. Platonov, ”On stability and dissipativity of some classes of complex systems”, Autom. Remote Control, 70:8 (2009), 1265-1280.

3. Applications to the Problem of Mobile Agent Deployment on a Line Segment

3.1. First order integrators

An important class of formation control problems is deployment of mobile agents on a line segment or given curve, see, for instance, [Wagner, Bruckstein, 1997], [Proskurnikov, Parsegov, 2016], [Martinez and F. Bullo, 2006], [Terushkin, Fridman, 2021]. Among potential applications of deployment control are sensor networks, patrolling, monitoring, coverage, partition, etc.

Consider a group of n mobile agents on a line. The agents are interpreted as numbered points with coordinates $x_i(t) \in \mathbb{R}$, $i = 1, \ldots, n$.

Denote $x(t) = (x_1(t), \ldots, x_n(t))^\top$.

Let the dynamics of agents be described by the equations

$$\dot{x}_i(t) = u_i, \qquad i = 1, \ldots, n. \tag{1}$$

Here u_i is a control input.

Assume that a segment $[a, b]$ of the line is given. We will consider the points a and b as static agents: $x_0(t) = a$, $x_{n+1}(t) = b$ for $t \geq 0$.

Problem: It is required to design decentralized control protocol ensuring the equidistant deployment of agents on the segment.

Hence, the corresponding closed-loop system should admit the asymptotically stable equilibrium position $\widehat{x} = (\widehat{x}_1, \ldots, \widehat{x}_n)^\top$, where

$$\widehat{x}_i = a + \frac{i}{n+1}(b - a), \quad i = 1, \ldots, n. \tag{2}$$

The averaging principle.

Assumption 1. Each agent receives information about the distances between itself and its nearest left and right neighbors.

In [Wagner, Bruckstein, 1997], [Proskurnikov, Parsegov, 2016], it was proved that, under Assumption 1, the control protocol

$$u_i = \frac{1}{2}(x_{i-1}(t) - x_i(t)) + \frac{1}{2}(x_{i+1}(t) - x_i(t)), \quad i = 1, \ldots, n, \tag{3}$$

ensures the equidistant agent deployment.

Remark. Here and in what follows neighbors are understood in terms of agents numbers.

An extension of the averaging principle [Aleksandrov, Fradkov, Semenov, 2020]:

Assumption 2. Eeach agent receives information about the distances between itself and one of its left neighbors and one of its right neighbors (not necessarily nearest neighbors). In addition, each agent knows how many agents are located between itself and the agent from which the signal is received, but it has no information about the total number of agents in the network.

Let

$$u_i = \beta_i x_{m_i}(t) + \gamma_i x_{l_i}(t) - x_i(t), \quad i = 1, \dots, n. \tag{4}$$

Here m_i and l_i are numbers of the left and right neighbors, respectively, from which the ith agent receives information, $0 \le m_i < i$, $i < l_i \le n+1$,

$$\beta_i = \frac{l_i - i}{l_i - m_i}, \quad \gamma_i = \frac{i - m_i}{l_i - m_i}, \quad i = 1, \dots, n. \tag{5}$$

Theorem 1. The system (1) with control protocol (4) admits the asymptotically stable equilibrium position (2).

To prove the theorem, with the aid of the substitution $y(t) = x(t) - \widehat{x}$, transform the closed-loop system to the form

$$\dot{y}(t) = \tilde{A} y(t). \tag{6}$$

Here $\tilde{A} = \left\{ \tilde{a}_{ij} \right\}_{i,j=1}^{n}$ is a constant matrix such that, for every $i \in \{1, \dots, n\}$, we have $\tilde{a}_{ii} = -1$,

$$\tilde{a}_{im_i} = \frac{l_i - i}{l_i - m_i} \text{ for } m_i \neq 0, \quad \tilde{a}_{il_i} = \frac{i - m_i}{l_i - m_i} \text{ for } l_i \neq n + 1,$$

and the remaining components of the ith row are equal to zero.

Let us note that $\tilde{A}$ is a Metzler matrix.

It was proved that if $\theta = (\theta_1, \ldots, \theta_n)^\top$, where

$$\theta_i = 1 + \frac{1}{2} + \ldots + \frac{1}{2^{i-1}}, \quad i = 1, \ldots, n, \tag{7}$$

then

$$\tilde{A}\theta < 0. \tag{8}$$

Delayed and switched interactions [Aleksandrov, Fradkov, Semenov, 2020]:

Assumption 3. Communications between agents can be switched on and off at any time instant. If the connection with the left (right) neighbor is lost, the agent selects some other left (right) neighbor.

Assumption 4. Each agent receives information from its neighbors with a constant nonnegative delay τ.

In this case we will use the following control protocol:

$$u_i = \Big(\beta_i^{(\sigma)}(x_{m_i^{(\sigma)}}(t-\tau) - x_i(t))$$

$$+\gamma_i^{(\sigma)}(x_{l_i^{(\sigma)}}(t-\tau) - x_i(t))\Big), \quad i = 1, \dots, n. \tag{9}$$

Theorem 2. The system (1) with control protocol (9) admits the equilibrium position (2) that is asymptotically stable for any nonnegative delay τ and for any admissible switching law.

A generalized protocol [Aleksandrov, Andriyanova, 2022]:

Consider an extension of the previous approach for constructing control protocols to the case where each agent use information from multiple left and multiple right neighbors.

Assumption 5. Each agent receives information about the distances between itself and some of its left neighbors and some of its right neighbors. In addition, each agent knows how many agents are located between itself and agents from which the signals are received.

Denote by N_{il} and N_{ir} the sets of left and right neighbors, respectively, from which the ith agent receives information, $i = 1, \ldots, n$. Let $N_i = N_{il} \cup N_{ir}$. Thus, each ith agent knows the distances $x_i(t) - x_j(t)$ for $j \in N_i$.

Let

$$u_i = \sum_{j \in N_i} a_{ij}(x_j(t) - x_i(t)), \quad i = 1, \dots, n, \tag{10}$$

where

$$a_{ij} = \frac{\delta_i}{(i-j)q_{il}} \quad \text{for} \quad j \in N_{il},$$

$$a_{ij} = \frac{\delta_i}{(j-i)q_{ir}} \quad \text{for} \quad j \in N_{ir},$$

q_{ir} and q_{il} are the numbers of elements of the sets N_{il} and N_{ir}, respectively, and the coefficients δ_i are defined from the conditions

$$\delta_i \left(\frac{1}{q_{il}} \sum_{j \in N_{il}} \frac{1}{i-j} + \frac{1}{q_{ir}} \sum_{j \in N_{ir}} \frac{1}{j-i} \right) = 1, \ i = 1, \dots, n. \tag{11}$$

Theorem 3. Let Assumption 5 be fulfilled. Then the system (1) closed by the control (10) admits the equilibrium position $\widehat{x}$ that is asymptotically stable.

Remark. This result was extended to the case of switched and delayed interactions.

Remark. In [Aleksandrov, Andriyanova, 2022], results of a numerical simulation were presented demonstrating that the use of more complete information about the distances to neighbors allows faster convergence of agents to the target positions.

3.2. Double integrators

[Aleksandrov, Fradkov, Semenov, IEEE TAC, 2022]

$$\ddot{x}_i(t) + a\dot{x}_i(t) = u_i, \quad i = 1, \ldots, n. \tag{12}$$

Here a is a real number (damping coefficient).

Assumption 6. Eeach agent receives information about the distances between itself and one of its left neighbors and one of its right neighbors (not necessarily nearest neighbors). In addition, each agent knows how many agents are located between itself and the agent from which the signal is received.

Assumption 7. Communications between agents can be switched on and off at any time instant. If the connection with the left (right) neighbor is lost, the agent selects some other left (right) neighbor.

Assumption 8. Each agent receives information from its neighbors with a constant nonnegative delay τ.

Let

$$u_i = k\Big(\Big(\beta_i^{(\sigma)}(x_{m_i^{(\sigma)}}(t-\tau) - x_i(t)) + \gamma_i^{(\sigma)}(x_{l_i^{(\sigma)}}(t-\tau) - x_i(t))\Big)\Big), \qquad (13)$$

$$i = 1, \dots, n,$$

where $k > 0$ is a control gain.

Theorem 4. For any given $a_0 > 0$, the control protocol (13) with the coefficient k satisfying the condition

$$a_0 \geq \sqrt{k} \qquad (14)$$

provides the asymptotic stability of the equilibrium position (2) of (12) for any $a \geq a_0$, any nonnegative delay τ and an arbitrary admissible switching law.

Remark. It should be noted that in this case the corresponding closed-loop system is not positive. To prove Theorem 4, the special substitution

$$z_i(t) = \dot{x}_i(t) + h(x_i(t) - \widehat{x}_i), \quad i = 1, \ldots, n, \tag{15}$$

is used, where h is a positive tuning parameter.

For a special choice of h, (15) transforms (12) to a positive system.

Remark. Moreover, in [Aleksandrov, Fradkov, Semenov, IEEE TAC, 2022], protocols based on the relative velocity measurements were proposed, as well.

3.4. The problem of nonuniform agent deployment

[Aleksandrov, Andriyanova, 2022]

In numerous applications, instead of the uniform distribution of agents, a prescribed nonuniform distribution is required. For instance, in the problem of coverage control for mobile sensors, a cost function is usually introduced to evaluate how well a given curve or domain is covered by a sensor network, and sensors should converge to a configuration which optimizes the cost function. In addition, an important class of formation control problems is synchronization of processes with respect to certain functions of phase coordinates (the problem of output consensus).

Consider the problem of nonlinearly-uniform (uniform with respect to a given nonlinear function) deployment of agents on the segment $[a, b]$. Let agent dynamics be described by the equations (1).

Let a continuous and strictly increasing function $f(\zeta)$ is given for $s \in (-\infty, +\infty)$. It is required to design a decentralized control protocol guaranteeing the agent convergence to the positions $\tilde{x}_i$ for which the corresponding points $f(\tilde{x}_i)$ are uniformly distributed on the segment $[f(a), f(b)]$. In particular, $f(\zeta)$ might be a cost function in the problem of coverage control for mobile sensors.

Assumption 11. Each ith agent at each time instant knows the values $f(x_i(t)) - f(x_j(t-\tau))$ for $j \in N_i^{(\omega)} = N_{il}^{(\omega)} \cup N_{ir}^{(\omega)}$, where $\tau =$ const ≥ 0, $\omega = \omega(t)$ is a switching law defining the order of network topology switching, $N_{il}^{(\omega)}$ and $N_{ir}^{(\omega)}$ are the sets of indices of left and right neighbors, respectively, from which the ith agent receives information at the instant t, $i = 1, \dots, n$. In addition, each agent knows how many agents are located between itself and agents from which the signals are received.

Let

$$a_{ij}^{(m)} = \frac{\delta_i^{(m)}}{(i-j)q_{il}^{(m)}} \quad \text{for} \quad j \in N_{il}^{(m)},$$

$$a_{ij}^{(m)} = \frac{\delta_i^{(m)}}{(j-i)q_{ir}^{(m)}} \quad \text{for} \quad j \in N_{ir}^{(m)},$$

$q_{il}^{(m)}$ and $q_{ir}^{(m)}$ are the numbers of elements of the sets $N_{il}^{(m)}$ and $N_{ir}^{(m)}$, respectively, and the coefficients $\delta_i^{(m)}$ are defined from the conditions

$$\delta_i^{(m)} \left(\frac{1}{q_{il}^{(m)}} \sum_{j \in N_{il}^{(m)}} \frac{1}{i-j} + \frac{1}{q_{ir}^{(m)}} \sum_{j \in N_{ir}^{(m)}} \frac{1}{j-i} \right) = 1,$$

$$i = 1, \dots, n, \quad m = 1, \dots, M.$$

Choose a control protocol in the form

$$u_i = \sum_{j \in N_i^{(\omega)}} a_{ij}^{(\omega)} (f(x_j(t-\tau)) - f(x_i(t))), \quad i = 1, \dots, n, \tag{16}$$

and consider the associated closed-loop system

$$\dot{x}_i(t) = \sum_{j \in N_i^{(\omega)}} a_{ij}^{(\omega)} (f(x_j(t-\tau)) - f(x_i(t))), \ i = 1, \dots, n. \tag{17}$$

Theorem 6. The system (17) admits the equilibrium position $\tilde{x}$ that is globally asymptotically stable for any $\tau \geq 0$ and arbitrary admissible switching signal.

Proof: Let $\tilde{x}_0 = a$, $\tilde{x}_{n+1} = b$. With the aid of the substitution $z_i(t) = x_i(t) - \tilde{x}_i$, $i = 1, \ldots, n$, transform (17) to the system

$$\dot{z}_i(t) = -h_i(z_i(t)) + \sum_{j \in N_i^{(\omega)},\, 1 \le j \le n} a_{ij}^{(\omega)} h_j(z_j(t - \tau)), \tag{18}$$

$$i = 1, \ldots, n.$$

Here $h_i(z_i(t)) = f(z_i(t) + \tilde{x}_i) - f(\tilde{x}_i)$, $i = 1, \ldots, n$.

It should be noted that functions $h_i(z_i)$ are strictly increasing for $z_i \in (-\infty, +\infty)$, $h_i(0) = 0$ and $z_i h_i(z_i) > 0$ for $z_i \neq 0$, $i = 1, \ldots, n$.

Hence, (18) is a switched positive Persidskii-type system with delay.

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